

# HAHN-BANACH TYPE EXTENSION THEOREMS ON $p$ -OPERATOR SPACES

JUNG-JIN LEE

**ABSTRACT.** Let  $V \subseteq W$  be two operator spaces. Arveson-Wittstock-Hahn-Banach theorem asserts that every completely contractive map  $\varphi : V \rightarrow \mathcal{B}(H)$  has a completely contractive extension  $\tilde{\varphi} : W \rightarrow \mathcal{B}(H)$ , where  $\mathcal{B}(H)$  denotes the space of all bounded operators from a Hilbert space  $H$  to itself. In this paper, we show that this is not in general true for  $p$ -operator spaces, that is, we show that there are  $p$ -operator spaces  $V \subseteq W$ , an  $SQ_p$  space  $E$ , and a  $p$ -completely contractive map  $\varphi : V \rightarrow \mathcal{B}(E)$  such that  $\varphi$  does not extend to a  $p$ -completely contractive map on  $W$ . Restricting  $E$  to  $L_p$  spaces, we also consider a condition on  $W$  under which every completely contractive map  $\varphi : V \rightarrow \mathcal{B}(L_p(\mu))$  has a completely contractive extension  $\tilde{\varphi} : W \rightarrow \mathcal{B}(L_p(\mu))$ .

## 1. INTRODUCTION TO $p$ -OPERATOR SPACES

Throughout this paper, we assume  $1 < p, p' < \infty$  with  $1/p + 1/p' = 1$ , unless stated otherwise. For a Banach space  $X$ , we denote by  $\mathbb{M}_{m,n}(X)$  the linear space of all  $m \times n$  matrices with entries in  $X$ . By  $\mathbb{M}_n(X)$ , we will denote  $\mathbb{M}_{n,n}(X)$ . When  $X = \mathbb{C}$ , we will simply use  $\mathbb{M}_{m,n}$  (respectively,  $\mathbb{M}_n$ ) for  $\mathbb{M}_{m,n}(\mathbb{C})$  (respectively,  $\mathbb{M}_n(\mathbb{C})$ ). For Banach spaces  $X$  and  $Y$ , we will denote by  $\mathcal{B}(X, Y)$  the space of all bounded linear operators from  $X$  to  $Y$ . We will also use  $\mathcal{B}(X)$  for  $\mathcal{B}(X, X)$ . The  $\ell_p$  direct sum of  $n$  copies of  $X$  will be denoted by  $\ell_p^n(X)$ .

**Definition 1.1.** Let  $SQ_p$  denote the collection of subspaces of quotients of  $L_p$  spaces. A Banach space  $X$  is called a *concrete  $p$ -operator space* if  $X$  is a closed subspace of  $\mathcal{B}(E)$  for some  $E \in SQ_p$ .

Let  $E \in SQ_p$ . For a concrete  $p$ -operator space  $X \subseteq \mathcal{B}(E)$  and for each  $n \in \mathbb{N}$ , define a norm  $\|\cdot\|_n$  on  $\mathbb{M}_n(X)$  by identifying  $\mathbb{M}_n(X)$  as a subspace of  $\mathcal{B}(\ell_p^n(E))$ , and let  $M_n(X)$  denote the corresponding normed space. The norms  $\|\cdot\|_n$  then satisfy

$\mathcal{D}_\infty$ : for  $u \in M_n(X)$  and  $v \in M_m(X)$ , we have  $\|u \oplus v\|_{M_{n+m}(X)} = \max\{\|u\|_n, \|v\|_m\}$ .

$\mathcal{M}_p$ : for  $u \in M_m(X)$ ,  $\alpha \in \mathbb{M}_{n,m}$ , and  $\beta \in \mathbb{M}_{m,n}$ , we have  $\|\alpha u \beta\|_n \leq \|\alpha\| \|u\|_m \|\beta\|$ , where  $\|\alpha\|$  is the norm of  $\alpha$  as a member of  $\mathcal{B}(\ell_p^m, \ell_p^n)$ , and similarly for  $\beta$ .

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When  $p = 2$ , these are Ruan's axioms and 2-operator spaces are simply operator spaces because the  $SQ_2$  spaces are exactly the same as Hilbert spaces.

As in operator spaces, we can also define abstract  $p$ -operator spaces.

**Definition 1.2.** An *abstract  $p$ -operator space* is a Banach space  $X$  together with a sequence of norms  $\|\cdot\|_n$  defined on  $M_n(X)$  satisfying the conditions  $\mathcal{D}_\infty$  and  $\mathcal{M}_p$  above.

Thanks to Ruan's representation theorem[Rua88], we do not distinguish between concrete and abstract operator spaces. Le Merdy showed that this remains true for  $p$ -operator spaces.

**Theorem 1.3.** [LeM96, Theorem 4.1] *An abstract  $p$ -operator space  $X$  can be isometrically embedded in  $\mathcal{B}(E)$  for some  $E \in SQ_p$  in such a way that the canonical norms on  $M_n(X)$  arising from this embedding agree with the given norms.*

**Example 1.4.**

- (a) Suppose  $E$  and  $F$  are  $SQ_p$  spaces and let  $L = E \oplus_p F$ , the  $\ell_p$  direct sum of  $E$  and  $F$ . Then  $L$  is also an  $SQ_p$  space [Her71, Proposition 5] and the mapping

$$x \mapsto \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$$

is an isometric embedding of  $\mathcal{B}(E, F)$  into  $\mathcal{B}(L)$ . Using this we can view  $\mathcal{B}(E, F)$  as a  $p$ -operator space. Note that  $M_n(\mathcal{B}(E, F))$  is isometrically isomorphic to  $\mathcal{B}(\ell_p^n(E), \ell_p^n(F))$ .

- (b) The identification  $L_p(\mu) = \mathcal{B}(\mathbb{C}, L_p(\mu)) \subseteq \mathcal{B}(\mathbb{C} \oplus_p L_p(\mu))$  gives a  $p$ -operator space structure on  $L_p(\mu)$  called the *column  $p$ -operator space structure* of  $L_p(\mu)$ , which we denote by  $L_p^c(\mu)$ . Similarly, the identification  $L_{p'}(\mu) = \mathcal{B}(L_p(\mu), \mathbb{C})$  gives rise to  $p$ -operator space structure on  $L_{p'}(\mu)$  which we denote by  $L_{p'}^r(\mu)$  and call the *row  $p$ -operator space structure* of  $L_{p'}(\mu)$ . In general, we can define  $E^c$  and  $(E')^r$  for any  $E \in SQ_p$ , where  $E'$  is the Banach dual space of  $E$ .

Note that a linear map  $u : X \rightarrow Y$  between  $p$ -operator spaces  $X$  and  $Y$  induces a map  $u_n : M_n(X) \rightarrow M_n(Y)$  by applying  $u$  entrywise. We say that  $u$  is  *$p$ -completely bounded* if  $\|u\|_{pcb} := \sup_n \|u_n\| < \infty$ . Similarly, we define  *$p$ -completely contractive*,  *$p$ -completely isometric*, and  *$p$ -completely quotient* maps. We write  $\mathcal{CB}_p(X, Y)$  for the space of all  $p$ -completely bounded maps from  $X$  into  $Y$ .

To turn the mapping space  $\mathcal{CB}_p(X, Y)$  between two  $p$ -operator spaces  $X$  and  $Y$  into a  $p$ -operator space, we define a norm on  $M_n(\mathcal{CB}_p(X, Y))$  by identifying this space with  $\mathcal{CB}_p(X, M_n(Y))$ . Using Le Merdy's theorem, one can show that  $\mathcal{CB}_p(X, Y)$  itself is a  $p$ -operator space. In particular, the  *$p$ -operator dual space* of  $X$  is defined to be  $\mathcal{CB}_p(X, \mathbb{C})$ . The next lemma by Daws shows that we may identify the Banach dual space  $X'$  of  $X$  with the  $p$ -operator dual space  $\mathcal{CB}_p(X, \mathbb{C})$  of  $X$ .

**Lemma 1.5.** [Daw10, Lemma 4.2] *Let  $X$  be a  $p$ -operator space, and let  $\varphi \in X'$ , the Banach dual of  $X$ . Then  $\varphi$  is  $p$ -completely bounded as a map to  $\mathbb{C}$ . Moreover,  $\|\varphi\|_{pcb} = \|\varphi\|$ .*

If  $E = L_p(\mu)$  for some measure  $\mu$  and  $X \subseteq \mathcal{B}(E) = \mathcal{B}(L_p(\mu))$ , then we say that  $X$  is a  $p$ -operator space on  $L_p$  space. These  $p$ -operator spaces are often easier to work with. For example, let  $\kappa_X : X \rightarrow X''$  denote the canonical inclusion from a  $p$ -operator space  $X$  into its second dual. Contrary to operator spaces,  $\kappa_X$  is *not* always  $p$ -completely isometric. Thanks to the following theorem by Daws, however, we can easily characterize those  $p$ -operator spaces with the property that the canonical inclusion is  $p$ -completely isometric.

**Proposition 1.6.** [Daw10, Proposition 4.4] *Let  $X$  be a  $p$ -operator space. Then  $\kappa_X$  is a  $p$ -complete contraction. Moreover,  $\kappa_X$  is a  $p$ -complete isometry if and only if  $X \subseteq \mathcal{B}(L_p(\mu))$   $p$ -completely isometrically for some measure  $\mu$ .*

## 2. NON-EXISTENCE OF $p$ -ARVESON-WITTSTOCK-HAHN-BANACH THEOREM

Let  $V \subseteq W$  be two operator spaces. Arveson-Wittstock-Hahn-Banach theorem asserts that every completely bounded map  $\varphi : V \rightarrow \mathcal{B}(H)$  has a completely bounded extension  $\tilde{\varphi} : W \rightarrow \mathcal{B}(H)$ , where  $H$  is a Hilbert space. For  $p$ -operator spaces, the following question naturally arises.

**Question 2.1.** Let  $V \subseteq W$  be  $p$ -operator spaces and  $E$  an  $SQ_p$  space. Does every  $p$ -completely bounded map  $\varphi : V \rightarrow \mathcal{B}(E)$  have a  $p$ -completely bounded extension  $\tilde{\varphi} : W \rightarrow \mathcal{B}(E)$ ?

To show that this question has a negative answer, let  $p \neq 2$ , and let  $E$  and  $L_p(\Omega)$  such that  $E$  is a Hilbert space embedding to  $L_p(\Omega)$ . The existence of such  $E$  and  $L_p(\Omega)$  is guaranteed by, for example, [DF93, Proposition 8.7]. Let  $J : E \hookrightarrow L_p(\Omega)$  denote the isometric embedding, then we can view  $E$  as a subspace of  $L_p(\Omega)$ .

**Lemma 2.2.** *Let  $J$  be as above. With  $p$ -operator space structure  $E^c$  and  $L_p(\Omega)^c$ ,  $J$  becomes a  $p$ -complete isometry.*

*Proof.* From Example 1.4, we note that  $M_n(E^c) \subseteq M_n(\mathcal{B}(\mathbb{C}, E)) = \mathcal{B}(\ell_p^n, \ell_p^n(E))$ . For  $[\xi_{ij}] \in M_n(E^c)$ , the norm is given by

$$\|[\xi_{ij}]\|^p = \sup \left\{ \sum_{i=1}^n \left\| \sum_{j=1}^n \lambda_j \xi_{ij} \right\|_E^p : \lambda_j \in \mathbb{C}, \sum_{j=1}^n |\lambda_j|^p \leq 1 \right\}.$$

Since  $J$  is an isometry,

$$\left\| J \left( \sum_{j=1}^n \lambda_j \xi_{ij} \right) \right\|_{L_p(\Omega)} = \left\| \sum_{j=1}^n \lambda_j \xi_{ij} \right\|_E$$

and it follows that

$$\begin{aligned}
\|J_n([\xi_{ij}])\|^p &= \sup \left\{ \sum_{i=1}^n \left\| \sum_{j=1}^n \lambda_j J(\xi_{ij}) \right\|_{L_p(\Omega)}^p : \lambda_j \in \mathbb{C}, \sum_{j=1}^n |\lambda_j|^p \leq 1 \right\} \\
&= \sup \left\{ \sum_{i=1}^n \left\| J \left( \sum_{j=1}^n \lambda_j \xi_{ij} \right) \right\|_{L_p(\Omega)}^p : \lambda_j \in \mathbb{C}, \sum_{j=1}^n |\lambda_j|^p \leq 1 \right\} \\
&= \|[\xi_{ij}]\|^p.
\end{aligned}$$

□

Let  $\tilde{E} = \mathbb{C} \oplus_p E$ . Let  $\pi : \tilde{E} \rightarrow E$  denote the projection from  $\tilde{E}$  onto  $E$  and define  $\varphi : E^c \rightarrow \mathcal{B}(\tilde{E})$  and  $\psi : \mathcal{B}(\tilde{E}) \rightarrow E^c$  by

$$\varphi(\xi) = T_\xi, \quad T_\xi(\lambda \oplus_p e) = 0 \oplus_p \lambda \xi, \quad \lambda \in \mathbb{C}, \quad e \in E$$

and

$$\psi(T) = \pi T(1 \oplus_p 0), \quad T \in \mathcal{B}(\tilde{E})$$

(see the diagram below).

$$\begin{array}{ccc}
L_p(\Omega)^c & & \\
\uparrow J & \searrow \tilde{\varphi} & \\
E^c & \xrightleftharpoons[\psi]{\varphi} & \mathcal{B}(\tilde{E})
\end{array}$$

It is then easy to check that  $\varphi$  and  $\psi$  are  $p$ -complete contractions with  $\psi \circ \varphi = id_{E^c}$ . Suppose that  $\varphi : E^c \rightarrow \mathcal{B}(\tilde{E})$  extends to  $\tilde{\varphi} : L_p(\Omega)^c \rightarrow \mathcal{B}(\tilde{E})$ . Define  $P : L_p(\Omega)^c \rightarrow E^c$  by  $P = \psi \circ \tilde{\varphi}$ , then it follows that  $P$  is a  $p$ -completely contractive projection onto  $E^c$ , meaning that  $E$  must be a 1-complemented subspace of  $L_p(\Omega)$ . This is, however, impossible, because it would imply that a Hilbert space  $E$  is isometrically isomorphic to some  $L_p$  space with  $p \neq 2$ .

### 3. A PREDUAL OF $\mathcal{CB}_p(V, M_n)$

In this section, we define a normed space structure on  $\mathbb{M}_n(V)$  whose Banach dual is isometrically isomorphic to  $\mathcal{CB}_p(V, M_n)$ .

**Lemma 3.1.** *Let  $1 < p, p' < \infty$  with  $1/p + 1/p' = 1$ . Let  $\lambda = \{\lambda_j\}_{1 \leq j \leq n}$  be a finite sequence in  $\mathbb{C}$ . Then*

$$\|\lambda\|_{\ell_p^n} \leq n^{1/p-1/p'} \cdot \|\lambda\|_{\ell_{p'}^n}.$$

*Proof.* There is nothing to prove if  $p = p' = 2$ . If  $p > p'$ , then  $\|\lambda\|_{\ell_p^n} \leq \|\lambda\|_{\ell_{p'}^n} \leq n^{1/p-1/p'} \cdot \|\lambda\|_{\ell_{p'}^n}$  since  $n^{1/p-1/p'} \geq 1$ . Finally, assume  $1 < p < p'$  and let  $q = \frac{p'}{p} > 1$  and let  $q'$  be the conjugate exponent to  $q$ . By Hölder's inequality,

$$\|\lambda\|_{\ell_p^n}^p \leq \left( \sum_{j=1}^n |\lambda_j|^{pq} \right)^{1/q} \cdot n^{1/q'} = \left( \sum_{j=1}^n |\lambda_j|^{p'} \right)^{p/p'} \cdot n^{1-p/p'}$$

and hence  $\|\lambda\|_{\ell_p^n} \leq n^{1/p-1/p'} \cdot \|\lambda\|_{\ell_{p'}^n}$ .  $\square$

**Lemma 3.2.** *Let  $\alpha = [\alpha_{ij}] \in \mathbb{M}_{n,r}$  and  $\beta = [\beta_{kl}] \in \mathbb{M}_{r,n}$ . Let  $1 < p, p' < \infty$  with  $1/p + 1/p' = 1$ . Then we have*

$$\|\alpha\|_{\mathcal{B}(\ell_p^r, \ell_p^n)} \leq \|\alpha\|_{p'} \cdot n^{1/p-1/p'} \quad \text{and} \quad \|\beta\|_{\mathcal{B}(\ell_p^n, \ell_p^r)} \leq \|\beta\|_p \cdot n^{1/p-1/p'},$$

where

$$\|\alpha\|_{p'} = \left( \sum_{i=1}^n \sum_{j=1}^r |\alpha_{ij}|^{p'} \right)^{1/p'} \quad \text{and} \quad \|\beta\|_p = \left( \sum_{k=1}^r \sum_{l=1}^n |\beta_{kl}|^p \right)^{1/p}.$$

*Proof.* Suppose  $\xi = \{\xi_j\}_{j=1}^r$  is a unit vector in  $\ell_p^r$ . For each  $i$ ,  $1 \leq i \leq n$ , let  $\eta_i = \left| \sum_{j=1}^r \alpha_{ij} \xi_j \right|$ , then by Hölder's inequality,  $\eta_i \leq \left( \sum_{j=1}^r |\alpha_{ij}|^{p'} \right)^{1/p'}$  and by Lemma 3.1,

$$\left( \sum_{i=1}^n \eta_i^p \right)^{1/p} \leq n^{1/p-1/p'} \cdot \left( \sum_{i=1}^n \eta_i^{p'} \right)^{1/p'} \leq n^{1/p-1/p'} \cdot \|\alpha\|_{p'}$$

and hence we get  $\|\alpha\|_{\mathcal{B}(\ell_p^r, \ell_p^n)} \leq n^{1/p-1/p'} \cdot \|\alpha\|_{p'}$ . To prove the second inequality, let  $\gamma = \beta^T \in \mathbb{M}_{n,r}$ , the transpose of  $\beta$ . Then by the argument above we have

$$\|\gamma\|_{\mathcal{B}(\ell_{p'}^r, \ell_{p'}^n)} \leq \|\gamma\|_p \cdot n^{1/p-1/p'}.$$

Since  $\|\gamma\|_{\mathcal{B}(\ell_{p'}^r, \ell_{p'}^n)} = \|\beta\|_{\mathcal{B}(\ell_p^n, \ell_p^r)}$  and  $\|\gamma\|_p = \|\beta\|_p$ , we get the desired inequality.  $\square$

Let  $V$  be a  $p$ -operator space. Fix  $n \in \mathbb{N}$  and define  $\|\cdot\|_{1,n} : \mathbb{M}_n(V) \rightarrow [0, \infty)$  by

$$(3.1) \quad \|v\|_{1,n} = \inf \{ \|\alpha\|_{p'} \|w\| \|\beta\|_p : r \in \mathbb{N}, \quad v = \alpha w \beta, \quad \alpha \in \mathbb{M}_{n,r}, \quad \beta \in \mathbb{M}_{r,n}, \quad w \in M_r(V) \},$$

where  $\|\cdot\|_{p'}$  and  $\|\cdot\|_p$  as in Lemma 3.2.

**Proposition 3.3.** *Suppose that  $V$  is a  $p$ -operator space and  $n \in \mathbb{N}$ . Then  $\|\cdot\|_{1,n}$  defines a norm on  $\mathbb{M}_n(V)$ .*

*Proof.* Suppose  $v_1, v_2 \in \mathbb{M}_n(V)$ . Let  $\epsilon > 0$ . For  $i = 1, 2$ , we can find  $\alpha_i, \beta_i$ , and  $w_i$  such that  $v_i = \alpha_i w_i \beta_i$  with  $\|w_i\| \leq 1$  and

$$(3.2) \quad \|\alpha_i\|_{p'} < (\|v_i\|_{1,n} + \epsilon)^{1/p'}, \quad \|\beta_i\|_p < (\|v_i\|_{1,n} + \epsilon)^{1/p}.$$

Let

$$\alpha = [\alpha_1 \ \alpha_2], \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \text{and} \quad w = \begin{bmatrix} w_1 & \\ & w_2 \end{bmatrix},$$

then  $\|\alpha\|_{p'}^{p'} = \|\alpha_1\|_{p'}^{p'} + \|\alpha_2\|_{p'}^{p'}$ ,  $\|\beta\|_p^p = \|\beta_1\|_p^p + \|\beta_2\|_p^p$ , and  $\|w\| \leq 1$ . Since  $v_1 + v_2 = \alpha w \beta$ , it follows that

$$\begin{aligned} \|v_1 + v_2\|_{1,n} &\leq \|\alpha\|_{p'} \|\beta\|_p \\ \text{(Young's inequality)} &\leq \frac{\|\alpha\|_{p'}^{p'}}{p'} + \frac{\|\beta\|_p^p}{p} \\ &= \frac{\|\alpha_1\|_{p'}^{p'} + \|\alpha_2\|_{p'}^{p'}}{p'} + \frac{\|\beta_1\|_p^p + \|\beta_2\|_p^p}{p} \\ \text{(by (3.2))} &< \frac{\|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\epsilon}{p'} + \frac{\|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\epsilon}{p} \\ &= \|v_1\|_{1,n} + \|v_2\|_{1,n} + 2\epsilon. \end{aligned}$$

Since  $\epsilon$  is arbitrary, we get  $\|v_1 + v_2\|_{1,n} \leq \|v_1\|_{1,n} + \|v_2\|_{1,n}$ .

For any  $c \in \mathbb{C}$ , if  $v = \alpha w \beta$ , then we have  $cv = \alpha(cw)\beta$  and hence  $\|cv\|_{1,n} \leq \|\alpha\|_{p'} |c| \|w\| \|\beta\|_p$ . Taking the infimum, we get

$$(3.3) \quad \|cv\|_{1,n} \leq |c| \|v\|_{1,n}.$$

When  $c \neq 0$ , replacing  $c$  by  $1/c$  and  $v$  by  $cv$  in (3.3) gives

$$(3.4) \quad |c| \|v\|_{1,n} \leq \|cv\|_{1,n},$$

so (3.3) together with (3.4) gives  $\|cv\|_{1,n} = |c| \|v\|_{1,n}$ , which is obviously true when  $c = 0$ .

Finally, suppose  $\|v\|_{1,n} = 0$ . To show that  $v = 0$ , it suffices to show that

$$(3.5) \quad \|v\| \leq n^{2|1/p-1/p'|} \cdot \|v\|_{1,n}.$$

Indeed, if  $v = \alpha w \beta$  with  $\alpha \in \mathbb{M}_{n,r}$ ,  $\beta \in \mathbb{M}_{r,n}$ , and  $w \in M_r(v)$ , then

$$\begin{aligned} \|v\| &\leq \|\alpha\| \|w\| \|\beta\| \\ \text{(by Lemma 3.2)} &\leq \|\alpha\|_{p'} \cdot n^{|1/p-1/p'|} \cdot \|w\| \cdot \|\beta\|_p \cdot n^{|1/p-1/p'|} \\ &= n^{2|1/p-1/p'|} \cdot \|\alpha\|_{p'} \cdot \|w\| \cdot \|\beta\|_p. \end{aligned}$$

Taking the infimum, (3.5) follows. □

For a  $p$ -operator space  $V$ , let  $\mathcal{T}_n(V)$  denote the normed space  $(\mathbb{M}_n(V), \|\cdot\|_{1,n})$ .

**Lemma 3.4.** *For a  $p$ -operator space  $V$ ,  $\mathcal{T}_n(V)' = M_n(V') = \mathcal{CB}_p(V, M_n)$  isometrically.*

*Proof.* The second isometric isomorphism comes from the definition of the  $p$ -operator space structure on  $V'$ . We follow the idea as in [ER00, §4.1]. Let  $f = [f_{ij}] \in M_n(V') = \mathcal{CB}_p(V, M_n)$ . Note that

$$\|f\| = \sup\{\|\langle f, \tilde{v} \rangle\| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1\}.$$

Let  $D_{n \times r}^p$  denote the closed unit ball of  $\ell_p^{n \times r}$ , then

$$\begin{aligned} \|f\| &= \sup\{|\langle \langle f, \tilde{v} \rangle \eta, \xi \rangle| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1, \eta \in D_{n \times r}^p, \xi \in D_{n \times r}^{p'}\} \\ &= \sup\left\{\left|\sum_{i,j,k,l} f_{ij}(\tilde{v}_{kl})\eta_{(j,l)}\xi_{(i,k)}\right| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1, \eta \in D_{n \times r}^p, \xi \in D_{n \times r}^{p'}\right\} \\ &= \sup\left\{\left|\sum_{i,j=1}^n \left\langle f_{ij}, \sum_{k,l=1}^r \xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)} \right\rangle\right| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1, \eta \in D_{n \times r}^p, \xi \in D_{n \times r}^{p'}\right\}. \end{aligned}$$

Note that  $\sum_{k,l=1}^r \xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)}$  is the  $(i,j)$ -entry of the matrix product  $\alpha\tilde{v}\beta$ , where

$$\alpha = \begin{bmatrix} \xi_{(1,1)} & \cdots & \xi_{(1,r)} \\ \vdots & \ddots & \vdots \\ \xi_{(n,1)} & \cdots & \xi_{(n,r)} \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} \eta_{(1,1)} & \cdots & \eta_{(n,1)} \\ \vdots & \ddots & \vdots \\ \beta_{(1,r)} & \cdots & \eta_{(n,r)} \end{bmatrix},$$

so

$$\begin{aligned} \|f\| &= \sup\left\{\left|\sum_{i,j=1}^n \langle f_{ij}, (\alpha\tilde{v}\beta)_{ij} \rangle\right| : \|\tilde{v}\| \leq 1, \|\alpha\|_{p'} \leq 1, \|\beta\|_p \leq 1\right\} \\ &= \sup\{|\langle f, v \rangle| : v = \alpha\tilde{v}\beta, \|\tilde{v}\| \leq 1, \|\alpha\|_{p'} \leq 1, \|\beta\|_p \leq 1\} \\ (3.6) \quad &= \sup\{|\langle f, v \rangle| : \|v\|_{1,n} \leq 1\}. \end{aligned}$$

Define the scalar pairing  $\Phi : M_n(V') \rightarrow \mathcal{T}_n(V)'$  by  $f \mapsto \langle f, \cdot \rangle$ , then from (3.6) it follows that  $\Phi$  is an isometric isomorphism.  $\square$

**Proposition 3.5.** *Let  $V \subseteq W$  be  $p$ -operator spaces such that the inclusion  $\mathcal{T}_n(V) \hookrightarrow \mathcal{T}_n(W)$  is isometric. Then every  $p$ -completely contractive map  $\varphi : V \rightarrow \mathcal{B}(L_p(\Omega))$  has a completely contractive extension  $\tilde{\varphi} : W \rightarrow \mathcal{B}(L_p(\Omega))$ .*

*Proof.* Following [ER00, Corollary 4.1.4, Theorem 4.1.5], it suffices to assume that  $\mathcal{B}(L_p(\Omega)) = \mathcal{B}(\ell_p^n) = M_n$ . If the inclusion  $i : \mathcal{T}_n(V) \hookrightarrow \mathcal{T}_n(W)$  is isometric, then by Lemma 3.4, the adjoint  $i' : \mathcal{CB}_p(W, M_n) \rightarrow \mathcal{CB}_p(V, M_n)$ , which is a restriction mapping, is an exact quotient mapping.  $\square$

4.  $\ell_p$ -POLAR DECOMPOSITION

Let  $V \subseteq W$  be  $p$ -operator spaces. By Proposition 3.5, if the inclusion  $\mathcal{T}_n(V) \hookrightarrow \mathcal{T}_n(W)$  is isometric, then every  $p$ -completely contractive map  $\varphi : V \rightarrow \mathcal{B}(L_p(\Omega))$  has a completely contractive extension  $\tilde{\varphi} : W \rightarrow \mathcal{B}(L_p(\Omega))$ . In this section, we consider a condition on  $W$  under which the inclusion  $\mathcal{T}_n(V) \hookrightarrow \mathcal{T}_n(W)$  is isometric. Recall that the vector  $p$ -norm of  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$  is defined by

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

If we identify  $\mathbb{M}_{r,n}$  with  $\mathcal{B}(\ell_2^n, \ell_2^r)$ , the space of all bounded linear operators from  $\ell_2^n$  to  $\ell_2^r$ , it is well known that every  $\beta \in \mathbb{M}_{r,n}$  with  $r \geq n$  has a *polar decomposition*, that is,  $\beta$  can be written as  $\beta = \tau\beta_0$ , where  $\tau \in \mathbb{M}_{r,n}$  has orthonormal columns, that is,  $\tau$  is an isometry, and  $\beta_0 \in \mathbb{M}_n$  is positive semidefinite [HJ90, §7.3]. For  $p \neq 2$  and  $r \geq n$ , regarding  $\mathbb{M}_{r,n}$  as  $\mathcal{B}(\ell_p^n, \ell_p^r)$ , the space of all bounded linear operators from  $\ell_p^n$  to  $\ell_p^r$ , we ask if there is an  $\ell_p$ -analogue of the polar decomposition. First of all, we need to define what we should mean by polar decomposition when  $p \neq 2$ , because, for example, if  $T : \ell_p^n \rightarrow \ell_p^n$ , then the adjoint  $T'$  is from  $\ell_{p'}^n$  to  $\ell_{p'}^n$ , where  $1/p + 1/p' = 1$ , and therefore  $T'T$  is not defined, which in turn means we lose the concept of positive (semi)definiteness. We use the definition below as a natural  $p$ -analogue of the polar decomposition.

**Definition 4.1.** Let  $r \geq n$ . We say that  $\beta \in \mathbb{M}_{r,n} = \mathcal{B}(\ell_p^n, \ell_p^r)$  is  $\ell_p$ -*polar decomposable* if there is an isometry  $\tau \in \mathbb{M}_{r,n}$  and an operator  $\beta_0 \in \mathbb{M}_n$  such that  $\beta = \tau\beta_0$ . In this case, we say that  $\beta = \tau\beta_0$  is an  $\ell_p$ -*polar decomposition* of  $\beta$ . The set of all full rank  $\ell_p$ -polar decomposable  $r \times n$  matrices is denoted by  $\mathbb{M}_{r,n}^{(p)}$ .

**Remark 4.2.**

- (a) If  $r < n$ , then there is no isometry in  $\mathbb{M}_{r,n} = \mathcal{B}(\ell_p^n, \ell_p^r)$  and hence we only consider the case  $r \geq n$  in Definition 4.1.
- (b) It is well known [HJ90, §0.4] that  $\text{rank } AB \leq \min\{\text{rank } A, \text{rank } B\}$  whenever  $AB$  is defined for matrices  $A$  and  $B$ , so if  $\beta = \tau\beta_0$  is an  $\ell_p$ -polar decomposition of a full rank  $r \times n$  matrix  $\beta$ , then

$$n = \text{rank } \beta \leq \min\{\text{rank } \tau, \text{rank } \beta_0\} \leq n$$

and it follows that  $\text{rank } \tau = \text{rank } \beta_0 = n$ . In particular,  $\beta_0$  is nonsingular.

- (c) If  $\beta = \tau\beta_0$  is an  $\ell_p$ -polar decomposition of  $\beta$ , then  $\|\beta\|_p = \|\beta_0\|_p$ , where  $\|\cdot\|_p$  is as in Lemma 3.2.

To give a characterization of  $\ell_p$ -polar decomposable matrices, we begin with a characterization of isometries from  $\ell_p^n$  to  $\ell_p^r$ . Recall that for a vector  $x = (x_1, \dots, x_m)$ , we define  $\text{supp } x$ , the *support* of  $x$ , by  $\text{supp } x = \{i : 1 \leq i \leq m, x_i \neq 0\}$ .



**Lemma 4.3.** *Let  $1 < p < \infty$ ,  $p \neq 2$ , and  $r \geq n$ . Then  $\tau : \ell_p^n \rightarrow \ell_p^r$  is an isometry if and only if the columns of  $\tau$  have mutually disjoint supports with each vector  $p$ -norm equal to 1.*

*Proof.* Let  $\tau_j = \begin{bmatrix} \tau_{1j} \\ \vdots \\ \tau_{rj} \end{bmatrix}$  denote the  $j^{\text{th}}$  column of an  $r \times n$  matrix  $\tau$ . If  $\tau_1, \dots, \tau_n$  have mutually disjoint supports with each  $p$ -norm equal to 1, then for any  $x = (x_1, \dots, x_n) \in \ell_p^n$ , we get

$$\begin{aligned} \|\tau x\|_p^p &= \sum_{i=1}^r \left| \sum_{j=1}^n \tau_{ij} x_j \right|^p = \sum_{k=1}^n \sum_{i \in \text{supp } \tau_k} \left| \sum_{j=1}^n \tau_{ij} x_j \right|^p \\ &= \sum_{k=1}^n \sum_{i \in \text{supp } \tau_k} |\tau_{ik} x_k|^p = \sum_{k=1}^n |x_k|^p \sum_{i \in \text{supp } \tau_k} |\tau_{ik}|^p \\ &= \|x\|_p^p. \end{aligned}$$

Conversely, suppose  $\tau : \ell_p^n \rightarrow \ell_p^r$  is an isometry. Since  $\tau_j = \tau e_j$  for each  $j$ , where  $e_j$  denotes the unit vector in  $\ell_p^n$  whose only non-zero entry is 1 at the  $j^{\text{th}}$  place, it follows that  $\tau_j$  is of norm 1. To show that columns of  $\tau$  have mutually disjoint supports, let  $j \neq k$  and consider  $e_j \pm e_k$  in  $\ell_p^n$ . Since  $\|e_j \pm e_k\|_p = 2^{1/p}$ , we get  $\|\tau_j \pm \tau_k\|_p^p = 2$  and the result follows from [Roy88, Lemma 15.7.23].  $\square$

**Remark 4.4.** The result above remains true when  $p = 1$ .

Let  $V$  be a  $p$ -operator space. For  $v \in \mathbb{M}_n(V)$ , we define

$$(4.1) \quad \|v\|_{2,n} = \inf \{ \|\alpha\|_{p'} \|w\| \|\beta\|_p : r \in \mathbb{N}, \quad v = \alpha w \beta, \quad \alpha^T \in \mathbb{M}_{r,n}^{(p')}, \quad \beta \in \mathbb{M}_{r,n}^{(p)}, \quad w \in M_r(V) \},$$

where  $\alpha^T$  denotes the transpose of  $\alpha$  and

$$\|\alpha\|_{p'} = \left( \sum_{i=1}^n \sum_{j=1}^r |\alpha_{ij}|^{p'} \right)^{1/p'} \quad \text{and} \quad \|\beta\|_p = \left( \sum_{k=1}^r \sum_{l=1}^n |\beta_{kl}|^p \right)^{1/p}.$$

**Proposition 4.5.** *Let  $V \subseteq W$  be  $p$ -operator spaces. If  $\|w\|_{2,n} = \|w\|_{1,n}$  for all  $w \in \mathbb{M}_n(W)$ , then the inclusion  $\mathcal{T}_n(V) \hookrightarrow \mathcal{T}_n(W)$  is isometric.*

*Proof.* Let  $v \in \mathbb{M}_n(V)$ . It is clear that  $\|v\|_{\mathcal{T}_n(W)} \leq \|v\|_{\mathcal{T}_n(V)}$ . Suppose  $\|v\|_{\mathcal{T}_n(W)} < 1$ , then by assumption, one can find  $r \in \mathbb{N}$ ,  $\alpha \in \mathbb{M}_{n,r}$ ,  $\beta \in \mathbb{M}_{r,n}$ , and  $w \in M_r(W)$  such that  $v = \alpha w \beta$ ,  $\alpha^T \in \mathbb{M}_{r,n}^{(p')}$ ,  $\beta \in \mathbb{M}_{r,n}^{(p)}$ ,  $\|\alpha\|_{p'} < 1$ ,  $\|w\| < 1$ , and  $\|\beta\|_p < 1$ . Let  $\beta = \tau \beta_0$  (respectively,  $\alpha^T = \sigma \alpha_0$ ) be  $\ell_p$ -(respectively,  $\ell_{p'}$ -) polar decomposition of  $\beta$  (respectively,  $\alpha^T$ ), and set  $\tilde{w} = \sigma^T w \tau$ , then  $\|\tilde{w}\|_{M_n(W)} < 1$ . Moreover, by Remark 4.2,  $\alpha_0$  and  $\beta_0$  are invertible and hence  $\tilde{w} = (\alpha_0^T)^{-1} v \beta_0^{-1} \in M_n(V)$ , giving that  $\|\tilde{w}\|_{M_n(V)} < 1$ . Since  $v = \alpha_0^T \tilde{w} \beta_0$ ,  $\|\alpha_0^T\|_{p'} = \|\alpha\|_{p'} < 1$ , and  $\|\beta_0\|_p = \|\beta\|_p < 1$  by Remark 4.2, it follows that  $\|v\|_{\mathcal{T}_n(V)} < 1$ .  $\square$

For any  $v \in \mathbb{M}_n(V)$ , it is clear that  $\|v\|_{1,n} \leq \|v\|_{2,n}$ . At this moment of writing, we do not know of any nontrivial example of  $p$ -operator space  $V$  such that  $\|\cdot\|_{1,n} = \|\cdot\|_{2,n}$ . It is not even clear whether  $\|\cdot\|_{2,n}$  defines a norm on  $\mathbb{M}_n(V)$  for some  $p$ -operator space  $V$  (see Remark 4.7). However, thanks to Lemma 4.3, we can give a characterization of  $\ell_p$ -polar decomposable matrices which may lead to finding a nontrivial example of  $p$ -operator spaces  $V$  such that  $\|v\|_{1,n} = \|v\|_{2,n}$  for all  $v \in \mathbb{M}_n(V)$ .

**Proposition 4.6.** *Let  $1 < p < \infty$ ,  $p \neq 2$ , and  $r \geq n$ . Then  $\beta = \begin{bmatrix} \text{---} & u_1 & \text{---} \\ & \vdots & \\ \text{---} & u_r & \text{---} \end{bmatrix} \in \mathbb{M}_{r,n} = \mathcal{B}(\ell_p^n, \ell_p^r)$  is  $\ell_p$ -polar decomposable if and only if there are  $u_{j_1}, u_{j_2}, \dots, u_{j_n}$ , not necessarily distinct, such that each  $u_i$  ( $1 \leq i \leq r$ ) is a scalar multiple of  $u_{j_k}$  for some  $k$ ,  $1 \leq k \leq n$ .*

*Proof.* Let  $\beta = \begin{bmatrix} \text{---} & u_1 & \text{---} \\ & \vdots & \\ \text{---} & u_r & \text{---} \end{bmatrix} \in \mathbb{M}_{r,n} = \mathcal{B}(\ell_p^n, \ell_p^r)$ . Suppose that there are  $u_{j_1}, u_{j_2}, \dots, u_{j_n}$  (not necessarily distinct) such that each  $u_i$  ( $1 \leq i \leq r$ ) is a scalar multiple of  $u_{j_k}$  for some  $k$ ,  $1 \leq k \leq n$ . Rearranging rows of  $\beta$  with an appropriate permutation if necessary, we may assume that  $1 = j_1 < j_2 < j_3 < \dots < j_n \leq r$  and that for  $i$  with  $j_k \leq i < j_{k+1}$ ,  $u_i = c_i u_{j_k}$  for some scalar  $c_i$ . For each  $k$ ,  $1 \leq k \leq n$ , we define  $\lambda_k = \left( \sum_{j_k \leq i < j_{k+1}} |c_i|^p \right)^{-1/p}$ . Note that  $\lambda_k$  is well defined since  $c_{j_k} = 1$ . Define  $\tau \in \mathbb{M}_{r,n}$  and  $\beta_0 \in \mathbb{M}_n$  by

$$\tau = \begin{bmatrix} c_1 \lambda_1 & 0 & 0 & \cdots & 0 \\ c_2 \lambda_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{j_2-1} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & c_{j_2} \lambda_2 & 0 & \cdots & 0 \\ 0 & c_{j_2+1} \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c_{j_3-1} \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{j_n} \lambda_n \\ 0 & 0 & 0 & \cdots & c_{j_n+1} \lambda_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_r \lambda_n \end{bmatrix} \quad \text{and} \quad \beta_0 = \begin{bmatrix} \text{---} & \frac{1}{\lambda_1} u_{j_1} & \text{---} \\ \text{---} & \frac{1}{\lambda_2} u_{j_2} & \text{---} \\ & \vdots & \\ \text{---} & \frac{1}{\lambda_n} u_{j_n} & \text{---} \end{bmatrix},$$

then by Lemma 4.3, it follows that  $\beta = \tau \beta_0$  is an  $\ell_p$ -polar decomposition of  $\beta$ .

Conversely, assume that  $\beta = \tau \beta_0$  is a  $p$ -polar decomposition of  $\beta$ . To exclude triviality, we may assume that  $\beta$  contains no rows of only zeros. Let  $\tau_k$  denote the  $k^{\text{th}}$  column of  $\tau$ . By Lemma 4.3,  $\text{supp } \tau_k \neq \emptyset$  so

we can pick  $j_k \in \text{supp } \tau_k$ . Moreover, for each  $i$ ,  $1 \leq i \leq r$ , there is exactly one  $k(i)$  such that  $i \in \text{supp } \tau_{k(i)}$  and it follows that  $u_i$  is a constant multiple of  $u_{j_{k(i)}}$ .  $\square$

**Remark 4.7.** Let  $v_1 \in \mathbb{M}_n(V)$  and  $v_2 \in \mathbb{M}_m(V)$  for some  $p$ -operator space  $V$ , then one can easily show that  $\|cv_1\|_{2,n} = |c|\|v_1\|_{2,n}$ . Moreover, the decomposition  $v_1 = \alpha_1^T w_1 \beta_1$  and  $v_2 = \alpha_2^T w_2 \beta_2$  gives

$$(4.2) \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha_1^T \\ \alpha_2^T \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix},$$

which, combined with Proposition 4.6, shows that  $\|v_1 \oplus v_2\|_{2,n+m} \leq \|v_1\|_{2,n} + \|v_2\|_{2,m}$ .

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MOUNT HOLYOKE COLLEGE, SOUTH HADLEY, MA 01075, USA  
*E-mail address*, Jung-Jin Lee: [jjlee@matholyoke.edu](mailto:jjlee@matholyoke.edu)